Computing c- and a-functions from Entanglement

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arXiv:2509.03259 [hep-th]

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- Quantum field theories flow under the RG from the UV to IR.
- The number of effective degrees of freedom decreases during the flow.
- Irreversibility of RG:
  - $(1+1)$D: c-function (Zamolodchikov 1986).
  - $(2+1)$D: F-function (Casini-Huerta 2012).
  - $(3+1)$D: a-function (Cardy 1988, Komargodski-Schwimmer 2011).
- These functions are monotonic along the RG flow.

- The entanglement entropy (EE) provides a natural probe of d.o.f.
- Can we compute $c-$ and $a-$functions directly from the entanglement entropy in specific QFTs?

- Consider a quantum mechanical system with many degrees of freedom.
- Assume it is in the ground state $|\Psi\rangle$ (pure state).
- The density matrix of the total system is $\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|$.
- The von Neumann entropy $\mathcal{S}_{\text{tot}} = -\mathrm{tr}\rho_{\text{tot}} \log \rho_{\text{tot}}$ vanishes.
- Now divide the total system into subsystems A and B and assume that B is inaccessible to A.
- Trace out the part B of the Hilbert space in order to obtain the reduced density matrix of A: $\rho_A = \mathrm{tr}_B \rho_{\text{tot}}$.
- The entropy $\mathcal{S}(A) = -\mathrm{tr}_A \rho_A \log \rho_A$ is a measure of the entanglement between A and B.

In quantum field theory:

- Define an entangling surface (e.g. a sphere of radius R).
- Divide space into two subsystems A and B.
- $\mathcal{S}(A)$ measures quantum correlations across the boundary ∂A.
- $\mathcal{S}(A)$ is dominated by short-distance (UV) modes near ∂A.

Generic features:

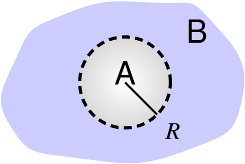
- *Area law:* $\mathcal{S}(A) \sim \mathrm{Area}(\partial A)/\epsilon^{d-1} + \ldots$, with cutoff ϵ.
- Divergent terms $\Rightarrow$ need subtractions or derivatives to extract universal, physical information.
- Universal subleading terms encode the *effective degrees of freedom* of the QFT.

Casini–Huerta, Phys.Lett. B600 (2004) 142-150

Casini–Teste–Torroba, Phys. Rev. Lett. 118, 261602 (2017)

Setup: Consider a QFT in its vacuum state and a spherical entangling surface of radius R.

- $\mathcal{S}(R)$ is the EE between A and B.
- $\Delta\mathcal{S}(R) = \mathcal{S}(R) - \mathcal{S}_{\mathrm{UV}}(R)$



From first principles:

- Poincaré invariance
- Strong subadditivity (SSA)
- Markov property of the vacuum

it has been argued that suitable combinations of derivatives of $\Delta\mathcal{S}(R)$ define monotonic functions that match CFT charges at the fixed points:

(1+1)D: $c(R) = R\,\Delta\mathcal{S}'(R)$, (3+1)D: $a(R) = R^2\,\Delta\mathcal{S}''(R) - R\,\Delta\mathcal{S}'(R)$.

$R \to 0$ (UV) : $c \to c_{\mathrm{UV}},\ a \to a_{\mathrm{UV}}$; $R \to \infty$ (IR) : $c \to c_{\mathrm{IR}},\ a \to a_{\mathrm{IR}}$.

Abate–Torroba, arXiv:2411.08961

Curved background (de Sitter):

- Replace Minkowski space by de Sitter space.
- Consider spherical regions in the static patch (bounded by a cosmological horizon at $R \sim H^{-1}$).
- Same ingredients: de Sitter invariance + SSA + Markov property.

The same expressions for $c(R)$ and $a(R)$ hold:

(1+1)D: $c(R) = R\,\Delta\mathcal{S}'(R)$, (3+1)D: $a(R) = R^2\,\Delta\mathcal{S}''(R) - R\,\Delta\mathcal{S}'(R)$.

In general, for $\Delta\mathcal{S}$ we subtract:

- The contribution of the UV fixed point
- Additional UV divergent terms, related to deformations of the conformal theory (curvature, mass).

Note: In the (3+1)D case, it was shown that $a_{\mathrm{UV}} > a_{\mathrm{IR}}$ but the monotonicity of $a(R)$ is not proven.

We focus on a **free massive scalar field theory** in flat background.

- The entropy can be computed explicitly (no interactions).
- Simplicity allows for numerically testing entropic c- and a-functions.

RG interpretation:

Turning on a mass μ triggers a trivial flow:

- In the deep UV the mass becomes irrelevant and we have one massless degree of freedom ($c_{\rm UV} = a_{\rm UV} = 1$).
- In the deep IR the massive field decouples and there are no degrees of freedom ($c_{\rm IR} = a_{\rm IR} = 0$).

Goal:

- Compute $\mathcal{S}(\mu, R)$ numerically for given theory.
- Use numerical derivatives to extract $c(R)$ in $(1+1)$D and $a(R)$ in $(3+1)$D.
- Check that they interpolate monotonically between 1 and 0.

$$(3+1)\text{D:} \quad \boxed{H \;=\; \frac{1}{2}\int d^3x\left[\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2\right]}$$

We expand in spherical harmonics and impose Dirichlet BC:

$$\phi(\mathbf{x}) \;=\; \sum_{\ell,m}\frac{\phi_{\ell m}(r)}{r}\,Y_{\ell m}(\Omega), \qquad \pi(\mathbf{x}) \;=\; \sum_{\ell,m}\frac{\pi_{\ell m}(r)}{r}\,Y_{\ell m}(\Omega),$$

$$H \;=\; \sum_{\ell,m} H_{\ell m}, \qquad H_{\ell m} \;=\; \frac{1}{2}\int_0^\infty dr\left[\pi_{\ell m}^2 + (\partial_r\phi_{\ell m})^2 + \left(\frac{\ell(\ell+1)}{r^2} + \mu^2\right)\phi_{\ell m}^2\right]$$

We discretize the radial coordinate $r_i = \epsilon i$ with $i = 1,\ldots,N$:

$$\phi_{\ell m}(r) \;\to\; \frac{1}{\sqrt{\epsilon}}\,\phi_{\ell m,i}, \quad \pi_{\ell m}(r) \;\to\; \frac{1}{\sqrt{\epsilon}}\,\pi_{\ell m,i},$$

$$H_{\ell m} = \frac{1}{2}\sum_{i=1}^{N}\left(\pi_{\ell m,i}^2 + \sum_{j=1}^{N}\phi_{\ell m,i}K_{ij}^{(\ell m)}\phi_{\ell m,j}\right)$$

Coupling matrix:

$$\boxed{K_{ij}^{(\ell m)} = \frac{1}{\epsilon^2}\left[\left(2+\mu^2\epsilon^2+\frac{\ell(\ell+1)}{i^2}\right)\delta_{ij}-\delta_{i,j+1}\;-\;\delta_{i+1,j}\right]}$$

The wave function that describes the modes of each (ℓ, m)-sector at their ground state is a Gaussian state:

$$\Psi(\phi_{\ell m}) = \left(\det\frac{\Omega}{\pi}\right)^{1/4} e^{-\frac{1}{2}\phi_{\ell m}^T\Omega\,\phi_{\ell m}},$$

where $\Omega = K^{1/2}$ and $\phi_{\ell m}^T = [\phi_{\ell m,1}, \ldots, \phi_{\ell m,N}]$.

We assume an entangling surface of radius $R = \left(n+\frac{1}{2}\right)\epsilon$:

- Subsystem A: oscillators 1 to n.
- Subsystem C: oscillators $n+1$ to N (complementary).

The entanglement entropy of Gaussian states can be calculated by making use of correlation functions. (Sorkin 2012)
For every ℓ, m:

- Define the $2N \times 2N$ matrix $\mathcal{M} = 2iJ\mathrm{Re}\,(M)$, where

$$M = \begin{pmatrix} \langle \phi_i \phi_j \rangle & \langle \phi_i \pi_j \rangle \\ \langle \phi_i \pi_j \rangle^T & \langle \pi_i \pi_j \rangle \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

- The $2n \times 2n$ block A of this matrix for subsystem A is

$$\mathcal{M}_A = i \begin{pmatrix} 0 & (\Omega)_A \\ -\left(\Omega^{-1}\right)_A & 0 \end{pmatrix}.$$

- Its eigenvalues come in pairs $\pm\lambda_i$ and satisfy $|\lambda_i| \geq 1$.
- The entanglement entropy can be computed as

$$\mathcal{S}_{\ell m} = \sum_{i=0}^{n} \left(\frac{\lambda_i + 1}{2} \log \frac{\lambda_i + 1}{2} - \frac{\lambda_i - 1}{2} \log \frac{\lambda_i - 1}{2} \right),$$

where the sum is performed only over the positive eigenvalues.

Lohmayer–Neuberger–Schwimmer-Theisen, Phys.Lett.B685:222-227,2010

(a) Finite-size effects (FSE)

- We introduce an IR cutoff $1/L = 1/(N\epsilon)$.
- We determine the finite-size effects for every ℓ up to 220.
- The FSE are captured by the expansion

$$\mathcal{S}_\ell(n,N,\mu) = \mathcal{S}_{\ell,\infty}(n,\mu) + \sum_{k=0}^{k_{\mathsf{max}}} \frac{\mathcal{S}_\ell^{(k)}(n,\mu)}{N^{(2\ell+2+k)}},$$

- We subtract the FSE by keeping $\mathcal{S}_{\ell,\infty}(n)$.

(b) Limit $\ell \to \infty$

- We study the truncated sum

$$\mathcal{S}_\infty(n, \mu; \ell_{\text{max}}) = \sum_{\ell=0}^{\ell_{\text{max}}} (2\ell + 1) \mathcal{S}_{\ell,\infty}(n, \mu)$$

  as a function of ℓ_{max} ($\ell_{\text{max}} \leq 10^6$).

- The sum behaves as

$$\mathcal{S}_\infty(n, \mu; \ell_{\text{max}}) = \mathcal{S}_\infty(n, \mu) + \sum_{i=1}^{i_{\text{max}}} \frac{1}{\ell_{\text{max}}^{2i}} \left(a_i(n, \mu) + b_i(n, \mu) \ln \ell_{\text{max}}\right)$$

- We obtain the limit $\ell_{\text{max}} \to \infty$ by keeping $\mathcal{S}_\infty(n, \mu)$.

(c) Dependence on $R = \left(n + \frac{1}{2}\right) \epsilon$ and μ (numerical fits)

Results agree with theoretical expressions.
($(1+1)$D $\equiv \ell = 0$ of $(3+1)$D)

(3+1)D entropy
(Srednicki 1993, Solodukhin 2008, Hertzberg-Wilczek 2011)

$$\boxed{\mathcal{S}(\mu, R) = c\frac{R^2}{\epsilon^2} - \frac{1}{90}\ln\frac{R}{\epsilon} + \frac{1}{6}\mu^2 R^2 \ln(\mu\epsilon) + \mathcal{S}_{\mathsf{fin}}(\mu R)}$$

Universal logarithmic term. Coefficient is related to the coefficient of the A-type conformal anomaly.

(1+1)D entropy
(Holzhey-Larsen-Wilczek 1994, Korepin 2004, Calabrese-Cardy 2004)

$$\boxed{\mathcal{S}(\mu, R) = \frac{1}{6}\ln\frac{R}{\epsilon} + \mathcal{S}_{\mathsf{fin}}(\mu R)}$$

Universal logarithmic term. Coefficient is proportional to the central charge ($c = 1$) of the theory.

- The expressions that give the $c-$ and $a-$functions through derivatives of $\mathcal{S}(A)$ eliminate the divergent terms.
- In the numerical procedure, there is no need to subtract UV terms.
- We make use of the independent variable $x = \mu R$ (only dimensionless parameter).
- We adopt a specific normalization so that $c_{\mathrm{UV}} = a_{\mathrm{UV}} = 1$ and $c_{\mathrm{IR}} = a_{\mathrm{IR}} = 0$, as x increases from 0 to ∞, independently of μ.

(1+1)D (c-function)

$$\boxed{c(x) = 6x\,\mathcal{S}'(x,\mu) = 1 + 6x\,\mathcal{S}'_{\text{fin}}(x)}$$

(3+1)D (a-function)

$$\boxed{a(x) = -45\left(x^2\,\mathcal{S}''(x,\mu) - x\,\mathcal{S}'(x,\mu)\right) = 1 - 45\left(x^2\mathcal{S}''_{\text{fin}}(x) - x\,\mathcal{S}'_{\text{fin}}(x)\right)}$$

- $x = \mu R \to 0$:

  Analytically we find

$$\boxed{\mathcal{S}(\mu R) = \frac{1}{6} \ln \frac{R}{\epsilon} + \frac{1}{6} \mu^2 R^2 \left[\ln(\mu R) + \gamma - \frac{5}{6} \right] + \cdots}$$

  which implies a smooth behaviour near 0:

$$\boxed{c(x) = 1 + 2x^2 \left[\ln x + \gamma - \tfrac{1}{3} \right] + \cdots} \qquad \boxed{\lim_{x \to 0} c(x) = 1}$$

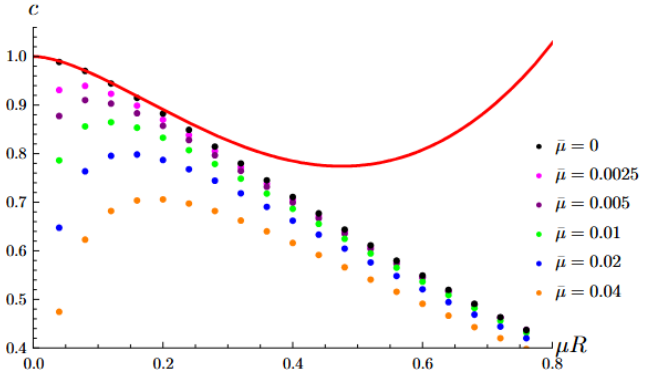
- $x = \mu R \to \infty$:

  We expect (deep IR, non-critical system)

$$\boxed{\mathcal{S}(\mu R) = \frac{1}{6} \ln \frac{R}{\epsilon} - \frac{1}{6} \ln(\mu R) = -\frac{1}{6} \ln(\mu \epsilon)} \qquad \boxed{\lim_{x \to \infty} c(x) = 0}$$

For $x \to 0$, we need $\bar{\mu} = \mu\epsilon \to 0$ to remove UV cutoff artifacts.
To do that:

- We compute $\mathcal{S}(R)$ for several $\bar{\mu}$.
- We perform numerical fits with $\bar{\mu}$ for fixed $x = \mu R$.
- Take limit $\bar{\mu} \to 0$.



The c-function interpolates between 1 and 0 monotonically.

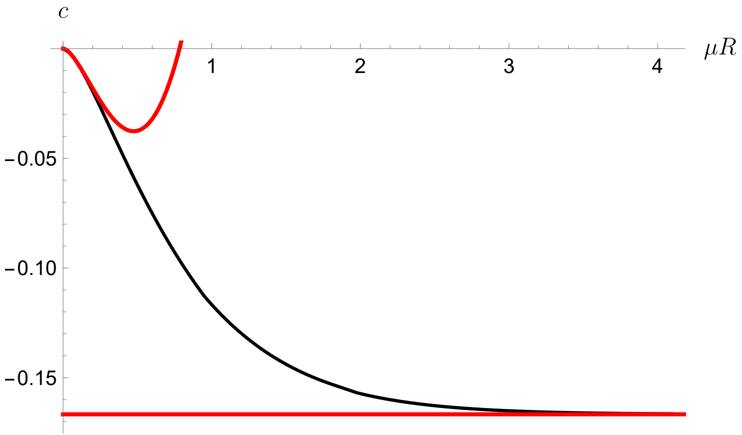
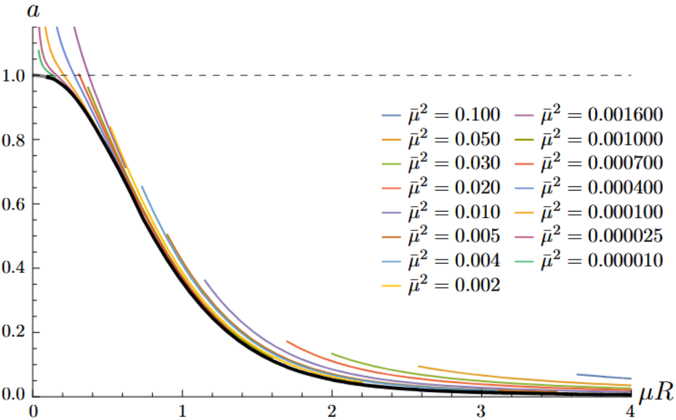


Figure: The c-function in terms of μR, along with the asymptotic expressions for small and large μR.

No analytic expressions for $x \to 0$ and $x \to \infty$. Same numerical procedure.

The a-function interpolates between 1 and 0 monotonically.



1. **F-function in $(2+1)$D**
   - Replace intervals/balls ($(1+1)$D/$(3+1)$D) with disks in $2+1$D QFT.
   - Define an entropic $F(R)$-function from $\mathcal{S}_{\text{disk}}(R)$ (related to a finite term).
2. **Extention to curved backgrounds (de Sitter):**
   - Repeat the numerical analysis for free massive scalar in dS, using appropriate coordinates.
   - Use entropic $c-$, $F-$ and $a-$functions in dS and test monotonicity beyond flat space.
3. **Interacting theories and non-trivial fixed points:**
   - Include interactions for non-trivial fixed points.
   - Compute entropic $c-$, $F-$, and a-functions.

Thank you!